

DISCRETE MATHEMATICAL STRUCTURES

Comprehensive Academic Question Bank

UNIT I: Logic, Proofs, and Set Theory

Syllabus Scope: Propositional logic, Propositional Equivalences, Predicates and Quantifiers, Nested Quantifiers, Rules of Inference, Introduction to Proofs. Sets: Venn Diagrams, Subsets, Size of a set, Power Sets, Cartesian Products, Set Operations.

A. Short Answer Questions (1 Mark Each)

- Q1. Define a proposition in mathematical logic and give an example.
- Q2. State the law of total contradiction or write down the truth table for a contradiction.
- Q3. What is a tautology?
- Q4. State the negation of the conditional statement $p \rightarrow q$.
- Q5. Define a predicate with a suitable example.
- Q6. Express the statement 'Every real number has an additive inverse' using quantifiers.
- Q7. What is the difference between an existential quantifier and a universal quantifier?
- Q8. State the rule of inference known as Modus Ponens.
- Q9. What is a direct proof?
- Q10. Define the term 'Counterexample'.
- Q11. If a set A has n elements, how many elements are in its Power Set $P(A)$?
- Q12. Define the Cartesian Product of two sets A and B.
- Q13. What does the cardinality of a set represent?
- Q14. Draw or describe a Venn diagram representing the symmetric difference of two sets.
- Q15. State De Morgan's Laws for set operations.
- Q16. True or False: If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
- Q17. What is an empty set, and what is its cardinality?
- Q18. Define a proper subset.
- Q19. Write the logical statement equivalent to the contrapositive of $p \rightarrow q$.
- Q20. What is the rule of inference called Modus Tollens?

B. Short Essay Questions (2 Marks Each)

- Q1. Construct the truth table for the proposition $(p \wedge q) \rightarrow (p \vee q)$.

- Q2.** Show that $\neg(p \rightarrow q)$ is logically equivalent to $p \wedge \neg q$ using truth tables.
- Q3.** Translate the following into logical expressions using quantifiers: 'Some students love mathematics but do not love physics.'
- Q4.** Distinguish between nested quantifiers $\forall x \exists y$ and $\exists y \forall x$ with an example.
- Q5.** Explain the method of proof by contradiction.
- Q6.** State the rules of universal instantiation and universal generalization.
- Q7.** If $A = \{1, 2\}$ and $B = \{a, b, c\}$, find $A \times B$ and its total number of elements.
- Q8.** Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ using set builder notation.
- Q9.** Define the symmetric difference of two sets A and B , and write its mathematical expression.
- Q10.** Given $A = \{x \in \mathbb{N} \mid x \text{ is a prime less than } 10\}$ and $B = \{x \in \mathbb{N} \mid x \text{ is an odd integer less than } 10\}$, find $A - B$ and $B - A$.
- Q11.** Explain the concept of mathematical induction in brief.
- Q12.** What is a bit-string representation of a set? Illustrate with a universal set of 5 elements.
- Q13.** Prove or disprove: The power set of the intersection of two sets is equal to the intersection of their power sets.
- Q14.** Explain the fallacy of affirming the conclusion with a symbolic example.
- Q15.** State the generalized Principle of Inclusion-Exclusion for three finite sets.

C. Medium Essay Questions (5 Marks Each)

- Q1.** Establish the logical equivalence of $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ using laws of logical equivalence (without using truth tables).
- Q2.** Determine the truth value of the statement $\exists x \forall y (x < y^2)$ if the domain for both variables consists of all real numbers. Explain your reasoning fully.
- Q3.** Show that the premises 'It is not sunny this afternoon and it is colder than yesterday,' 'We will go swimming only if it is sunny,' 'If we do not go swimming, then we will take a canoe trip,' and 'If we take a canoe trip, then we will be home by sunset' lead to the conclusion 'We will be home by sunset' using rules of inference.
- Q4.** Prove that if n is an integer and $n^3 + 5$ is odd, then n is even, using a proof by contraposition.
- Q5.** Prove that the square root of 2 is an irrational number using an indirect proof mechanism.
- Q6.** Let A , B , and C be arbitrary sets. Prove that $(A - B) - C = A - (B \cup C)$.
- Q7.** Among 120 computer science students, 65 know C++, 45 know Java, and 42 know Python. Further, 20 know both C++ and Java, 25 know both C++ and Python, and 15 know Java and Python. If 8 students know all three languages, find how many students know none of these languages.
- Q8.** Prove that if A and B are finite sets, then $|A \cup B| = |A| + |B| - |A \cap B|$.
- Q9.** Write a short technical note on the classification of proofs, explaining constructive vs. non-constructive existential proofs with instances.
- Q10.** Verify whether the following argument is valid: 'If I study, I will pass the exam. If I do not play games, I will study. I failed the exam. Therefore, I played games.'

D. Long Essay / Comprehensive Proofs (8 Marks Each)

- Q1.** Examine the foundational structures of propositional logic. Define tautology, contradiction, and contingency. Prove step-by-step using both truth tables and logical equivalence laws that $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$ is a tautology.
- Q2.** Elaborate thoroughly on the rules of inference for quantified statements. Analyze the validity of the following argument step-by-step, explicitly identifying which rule of inference is applied at each stage: 'Every student in this class understands discrete math. John is a student in this class. Therefore, John understands discrete math.'
- Q3.** Provide a thorough overview of mathematical proofs. Contrast direct proofs, proofs by contraposition, and proofs by contradiction. Apply each of these strategies to prove the statement: 'If $3n + 2$ is an odd integer, then n is an odd integer.'
- Q4.** Formulate a complete algebraic analysis of set operations. Prove the generalized absorption laws, distributive laws, and De Morgan's laws for any three arbitrary sets A , B , and C using subset inclusion arguments (double containment proofs).
- Q5.** In a market survey of 500 consumers regarding three products X , Y , and Z , the following data was gathered: 220 preferred X , 250 preferred Y , 230 preferred Z , 100 preferred X and Y , 110 preferred Y and Z , 90 preferred X and Z , and 30 preferred all three. Construct a comprehensive Venn diagram analyzing this data and solve for: (i) number of consumers who prefer product X only, (ii) consumers preferring exactly two products, and (iii) consumers preferring none of the products.

UNIT II: Functions, Sequences, Recurrence Relations, and Summations

Syllabus Scope: Functions: One-to-One (Injective) and Onto (Surjective) Functions, Inverse Functions, Compositions of Functions, Partial Functions. Sequences, Recurrence Relations, Summations.

A. Short Answer Questions (1 Mark Each)

- Q1.** Define a one-to-one (injective) function.
- Q2.** What condition must a function satisfy to possess an inverse?
- Q3.** Define an onto (surjective) function.
- Q4.** What is a bijection?
- Q5.** Define a partial function.
- Q6.** Given $f(x) = 2x + 3$, find its inverse function $f^{-1}(x)$.
- Q7.** What is the domain and codomain of a sequence?
- Q8.** Define a geometric progression sequence.
- Q9.** What is a recurrence relation?
- Q10.** State the closed-form formula for the summation of the first n positive integers.
- Q11.** What is the value of the summation $\sum$ (from $k=1$ to n) of a constant c ?
- Q12.** Define the composition of two functions f and g , denoted by $f \circ g$.

- Q13.** What is the floor function of -2.5?
- Q14.** What is the ceiling function of 3.1?
- Q15.** State the recurrence relation that generates the Fibonacci sequence.
- Q16.** What is the characteristic equation of a linear homogeneous recurrence relation of degree 2?
- Q17.** Define a sequence in discrete mathematics.
- Q18.** What is the sum of an infinite geometric series with a common ratio r where $|r| < 1$?
- Q19.** What differentiates a total function from a partial function?
- Q20.** Write the general term of an arithmetic progression with first term a and common difference d .

B. Short Essay Questions (2 Marks Each)

- Q1.** Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = 3x^2 + 1$. Determine whether f is injective.
- Q2.** Let $g: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $g(x) = x + 5$. Prove that g is surjective.
- Q3.** Find the composition $f \circ g$ if $f(x) = x^2$ and $g(x) = x + 2$ over the domain of real numbers.
- Q4.** Find the value of the summation $\sum_{k=1}^5 (k^2 - 1)$.
- Q5.** Determine whether the sequence $a_n = 3^n$ is a solution to the recurrence relation $a_n = 2a_{n-1} + 3a_{n-2}$.
- Q6.** Define a partial function and provide a practical scenario in computer science where partial functions occur.
- Q7.** Find the closed form for the summation $\sum_{k=0}^n 2^k$.
- Q8.** Find the inverse of the bijective function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = (x - 7) / 3$.
- Q9.** State the initial conditions necessary to uniquely solve a second-order linear recurrence relation.
- Q10.** Show that the ceiling function $\lceil x + n \rceil = \lceil x \rceil + n$ for any real number x and integer n .
- Q11.** If $f(x) = ax + b$ and $g(x) = cx + d$, find conditions under which $f \circ g = g \circ f$.
- Q12.** Identify the first four terms of the sequence defined recursively by $a_0 = 2$, $a_n = 2a_{n-1} + 1$.
- Q13.** Express the repeating decimal $0.7777\ldots$ as a ratio of two integers using a geometric series.
- Q14.** What is the difference between a homogeneous and non-homogeneous recurrence relation?
- Q15.** Evaluate the double summation $\sum_{i=1}^3 \sum_{j=1}^2 (i * j)$.

C. Medium Essay Questions (5 Marks Each)

- Q1.** Let f and g be functions from the set of integers to the set of integers such that $f(x) = 2x + 1$ and $g(x) = 3x + 4$. Determine formulas for $f \circ g$ and $g \circ f$. Is composition commutative in this case?
- Q2.** Prove that if $f: A \rightarrow B$ and $g: B \rightarrow C$ are both one-to-one functions, then the composition $g \circ f$ is also a one-to-one function.
- Q3.** Find the explicit closed-form solution to the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, given the initial conditions $a_0 = 1$ and $a_1 = 4$.
- Q4.** Solve the linear homogeneous recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with initial conditions $a_0 = 2$ and $a_1 = 3$.
- Q5.** Find the closed form for the summation $\sum_{k=1}^n k(k+1)$ using standard summation properties.
- Q6.** Let $f: A \rightarrow B$ be a function. Prove that f is onto if and only if for every subset Y of B , the image of its preimage satisfies $f(f^{-1}(Y)) = Y$.

- Q7.** Show using mathematical induction that the n th term of the sequence defined by $a_0 = 1$, $a_n = 2a_{n-1} + 1$ is given by $a_n = 2^{n+1} - 1$.
- Q8.** Discuss partial functions. Provide their formal definition, mapping behavior, and detail how they are transformed into total functions via error states or undefined tokens.
- Q9.** Evaluate the summation $\sum$ (from $k=1$ to n) $k \cdot 2^k$ using the perturbation method or summation by parts.
- Q10.** Explain the concept of sequences and their limits. Find the general formula for the sequence $1/2, 2/3, 3/4, 4/5, \dots$ and determine its behavior as n approaches infinity.

D. Long Essay / Comprehensive Proofs (8 Marks Each)

- Q1.** Analyze bijective functions and functional composition in detail. Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be two functions. Prove rigorously that: (i) If both f and g are bijections, then the composition $g \circ f$ is a bijection, and (ii) $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
- Q2.** Formulate a complete framework for solving second-order linear homogeneous recurrence relations with constant coefficients. Detail the three cases based on the discriminant of the characteristic equation (distinct real roots, equal real roots, complex roots) and solve the relation: $a_n = 4a_{n-1} - 4a_{n-2}$ with initial criteria $a_0 = 1$, $a_1 = 4$.
- Q3.** Write a comprehensive dissertation on summations and their applications in algorithm runtime analysis. Prove the closed forms for $\sum k$, $\sum k^2$, and $\sum k^3$ using inductive techniques, and explain how these summations are applied to bound the time complexity of nested loops.
- Q4.** Investigate non-homogeneous recurrence relations. Describe the method of undetermined coefficients for finding particular solutions. Solve the non-homogeneous relation $a_n = 3a_{n-1} + 2n$ given that $a_0 = 1$.
- Q5.** Define functions, sequences, and partial mappings. Let f be a function mapping from a real domain to a real codomain. Prove that any function f can be decomposed into an even function and an odd function. Provide an illustrative design for $f(x) = e^x$, deriving the hyperbolic functions as a result.

UNIT III: Counting Techniques and Relations

Syllabus Scope: Counting: Basics of Counting, Pigeonhole Principle, Permutations and Combinations, Binomial Coefficients and Identities. Relations: Relations and their Properties, n -ary Relations and Applications, Representing Relations, Closure of Relations, Equivalence Relations, Partial Orderings.

A. Short Answer Questions (1 Mark Each)

- Q1.** State the product rule of counting.
- Q2.** State the sum rule of counting.
- Q3.** What is the Pigeonhole Principle?
- Q4.** Define a permutation of n objects taken r at a time.
- Q5.** Write down the formula for the number of combinations of n objects taken r at a time.
- Q6.** State the Binomial Theorem formula.
- Q7.** What is Pascal's Identity for binomial coefficients?
- Q8.** Define a binary relation from a set A to a set B .

- Q9. What is a reflexive relation?
- Q10. When is a relation said to be symmetric?
- Q11. Define a transitive relation.
- Q12. What is an equivalence relation?
- Q13. What three properties define a partial ordering?
- Q14. What is a poset?
- Q15. Define the reflexive closure of a relation R.
- Q16. What is a directed graph representation of a relation?
- Q17. What is an n-ary relation?
- Q18. Define an equivalence class.
- Q19. State the value of $C(n, 0) + C(n, 1) + \dots + C(n, n)$.
- Q20. What is the total number of binary relations possible on a set with n elements?

B. Short Essay Questions (2 Marks Each)

- Q1. How many bit strings of length 8 either start with a 1 or end with 00?
- Q2. Show that among any group of 8 people, at least two must have been born on the same day of the week.
- Q3. Find the number of ways to arrange the letters of the word 'MATHEMATICS'.
- Q4. Prove that $C(n, r) = C(n, n - r)$ algebraically.
- Q5. Let R be a relation on the set of integers such that aRb if and only if $a - b$ is an even integer. Is R symmetric? Why?
- Q6. Give an example of a relation that is symmetric and transitive but not reflexive.
- Q7. Represent the relation $R = \{(1,1), (1,2), (2,3)\}$ on the set $\{1, 2, 3\}$ using a zero-one matrix.
- Q8. What is the symmetric closure of the relation $R = \{(1,2), (2,3), (3,1)\}$?
- Q9. Explain the concept of an n-ary relation and its primary application in relational databases.
- Q10. What is a Hasse diagram? What does it represent?
- Q11. Find the coefficient of $x^3 y^4$ in the expansion of $(x + y)^7$.
- Q12. How many integers between 1 and 100 are divisible by either 3 or 5?
- Q13. Define anti-symmetric property of a relation with a mathematical statement.
- Q14. Find the transitive closure of $R = \{(1,2), (2,3)\}$ on the set $\{1, 2, 3\}$.
- Q15. What is a total ordering? How does it differ from a partial ordering?

C. Medium Essay Questions (5 Marks Each)

- Q1. State and prove the Generalized Pigeonhole Principle. Show that in any group of 100 people, at least 9 were born in the same month.
- Q2. Prove Pascal's identity $C(n+1, k) = C(n, k-1) + C(n, k)$ using a combinatorial argument.
- Q3. Find the number of integer solutions to the equation $x_1 + x_2 + x_3 + x_4 = 17$, where $x_i \geq 0$ for all i .
- Q4. Let R be a relation on the set of ordered pairs of positive integers such that $(a,b)R(c,d)$ if and only if $ad = bc$. Prove that R is an equivalence relation.

- Q5.** Find the transitive closure of the relation $R = \{(1,2), (2,1), (2,3), (3,4)\}$ on the set $\{1, 2, 3, 4\}$ using Warshall's Algorithm. Show intermediate matrices.
- Q6.** Let R be a relation on the set of real numbers such that xRy if and only if $x - y$ is an integer. Prove that R is an equivalence relation and describe its equivalence classes.
- Q7.** Draw the Hasse diagram for the 'divides' relation on the set $A = \{1, 2, 3, 4, 6, 8, 12, 24\}$.
- Q8.** Prove using algebraic identities that $\sum_{k=0}^n (-1)^k C(n, k) = 0$.
- Q9.** Discuss the matrix representation of relations. Explain how reflexivity, symmetry, and transitivity can be determined directly from a relation's zero-one matrix.
- Q10.** How many ways are there to distribute 10 identical apples among 4 distinct children such that each child receives at least one apple?

D. Long Essay / Comprehensive Proofs (8 Marks Each)

- Q1.** Write a comprehensive treatise on combinatorial counting methodologies. Define and differentiate permutations and combinations with and without repetitions. Solve: (i) find the number of ways to select 5 items from a batch of 15 identical items with unlimited repetitions, and (ii) find the number of arrangements of the word 'COMBINATORICS' where vowels must never appear together.
- Q2.** Explain the theory of equivalence relations and partitions. Prove that an equivalence relation R on a set A partitions the set A into disjoint equivalence classes whose union equals A . Conversely, show that any partition of a set defines a unique equivalence relation.
- Q3.** Examine the closure of relations. Define reflexive, symmetric, and transitive closures. Provide a deep analysis of Warshall's Algorithm for computing transitive closures. Prove its correctness and apply it manually to find the transitive closure of the relation $R = \{(a,b), (b,c), (c,a), (c,d)\}$ on the set $\{a, b, c, d\}$.
- Q4.** Investigate Partially Ordered Sets (Posets) and Lattices. Define posets, maximal elements, minimal elements, greatest lower bounds (GLB), and least upper bounds (LUB). Draw the Hasse diagram for the power set of $\{a, b, c\}$ ordered by subset inclusion ($\subseteq$), and determine whether it forms a lattice, proving your conclusion.
- Q5.** State and prove the Principle of Inclusion-Exclusion for n sets. Apply this principle to find the number of primes less than or equal to 100 by setting up appropriate sets of composites divisible by 2, 3, 5, and 7, showing full arithmetic derivation steps.

UNIT IV: Graph Theory, Advanced Graph Algorithms, and Trees

Syllabus Scope: Graph Terminology and Special Types of Graphs, Bipartite Graphs, Representing Graphs, Isomorphism of Graphs, Euler and Hamilton Paths, Shortest Path Problems: Dijkstra's Algorithm, Traveling Salesperson Problem, Planar Graphs, Graph Coloring. Tree Traversal, Minimum Spanning Trees.

A. Short Answer Questions (1 Mark Each)

- Q1.** Define a simple graph.
- Q2.** What is the Handshaking Theorem equation?
- Q3.** Define a complete graph K_n .
- Q4.** What is a bipartite graph?

- Q5.** State the condition under which two graphs are called isomorphic.
- Q6.** What is an Euler path in a graph?
- Q7.** What is a Hamilton circuit?
- Q8.** Define a planar graph.
- Q9.** State Euler's formula for connected planar graphs connecting vertices, edges, and faces.
- Q10.** What is the chromatic number of a graph?
- Q11.** Name an algorithm used to find the shortest path in a weighted graph.
- Q12.** What is the Traveling Salesperson Problem (TSP)?
- Q13.** Define a tree in graph theory.
- Q14.** What is a minimum spanning tree?
- Q15.** List the three standard tree traversal strategies.
- Q16.** How many edges are present in a tree with n vertices?
- Q17.** What is a regular graph?
- Q18.** Define the degree of a vertex in an undirected graph.
- Q19.** What is an adjacency matrix representation of a graph?
- Q20.** What is the chromatic number of any bipartite graph with at least one edge?

B. Short Essay Questions (2 Marks Each)

- Q1.** Can an undirected graph have an odd number of vertices of odd degree? Explain using the Handshaking Theorem.
- Q2.** Show that the complete graph K_5 is non-planar.
- Q3.** State the necessary and sufficient condition for a connected graph to possess an Euler circuit.
- Q4.** Define graph isomorphism and list two invariant properties shared by isomorphic graphs.
- Q5.** What is the difference between an adjacency matrix and an incidence matrix?
- Q6.** Explain the concept of graph coloring and its practical link to exam scheduling problems.
- Q7.** Differentiate between an Euler path and a Hamilton path.
- Q8.** Give an example of a connected graph that has a Hamilton path but no Hamilton circuit.
- Q9.** Define a tree and list two alternative characterizations of a tree structure.
- Q10.** What is a rooted tree? Explain the terms parent, child, and leaf.
- Q11.** Show that the complete bipartite graph $K_{\{3,3\}}$ is non-planar.
- Q12.** Explain the greedy choice property underlying Kruskal's algorithm.
- Q13.** State the Four Color Theorem for planar graphs.
- Q14.** Sketch a small bipartite graph and provide its partition sets.
- Q15.** What is the runtime complexity of Dijkstra's algorithm using a binary heap?

C. Medium Essay Questions (5 Marks Each)

- Q1.** Verify if two graphs are isomorphic by examining invariants. Given two graphs G and H with 5 vertices and 6 edges each, demonstrate a step-by-step check using adjacency matrices or structural invariants.

- Q2.** Prove that a simple graph is bipartite if and only if it is possible to color its vertices using two colors such that no two adjacent vertices share the same color.
- Q3.** State Dijkstra's Algorithm for finding single-source shortest paths. Illustrate its step-by-step vertex relaxation on a 4-vertex weighted directed graph.
- Q4.** Explain planar graphs. Prove that for any connected planar simple graph with $v \geq 3$ vertices and e edges, the inequality $e \leq 3v - 6$ holds.
- Q5.** Discuss the Traveling Salesperson Problem (TSP). Detail its formulation, classification as NP-hard, and explain a simple heuristic approach to approximate its solution.
- Q6.** Perform Preorder, Inorder, and Postorder traversals on a given binary tree structure. Sketch the tree and list the resulting vertex sequences clearly.
- Q7.** Describe Prim's algorithm for constructing a Minimum Spanning Tree (MST). Explain how it builds the tree vertex-by-vertex compared to Kruskal's edge-by-edge approach.
- Q8.** Prove that a connected graph with n vertices is a tree if and only if it has exactly $n - 1$ edges.
- Q9.** Explain the Breadth-First Search (BFS) algorithm for graph traversal, outlining its pseudocode and its application in finding unweighted shortest paths.
- Q10.** Explain Depth-First Search (DFS) and show how it detects cycles in a directed graph using vertex coloring states (White, Gray, Black).

D. Long Essay / Comprehensive Proofs (8 Marks Each)

- Q1.** Formulate a complete technical analysis of Eulerian and Hamiltonian graphs. State and prove Dirac's Theorem and Ore's Theorem regarding the existence of Hamilton circuits in simple graphs. Contrast the structural requirements of Euler circuits vs. Hamilton circuits.
- Q2.** Write an exhaustive essay on planar graphs and graph coloring. State and prove Euler's Formula ($V - E + F = 2$) for connected planar graphs. Use this formula to show that K_5 and $K_{3,3}$ are non-planar, and discuss the chromatic numbering of planar networks via the Five Color Theorem proof structure.
- Q3.** Design a complete tracing of shortest path and minimum spanning tree algorithms. Given a complex 6-vertex weighted graph with designated edge weights, apply Dijkstra's Algorithm from a source vertex and Kruskal's Algorithm to find the MST. Show step-by-step edge selections, cost tables, and structural cuts.
- Q4.** Examine trees and tree traversals. Define an m -ary tree, full binary tree, and balanced tree. Prove that a full m -ary tree with i internal vertices has $n = mi + 1$ total vertices. Write recursive algorithms for Inorder, Preorder, and Postorder traversals and show their execution trace on a 4-level deep expression tree.
- Q5.** Investigate graph representations and search mechanics. Compare adjacency matrices, adjacency lists, and incidence matrices regarding space and time complexity for standard operations. Provide complete algorithmic frameworks for Breadth-First Search (BFS) and Depth-First Search (DFS), and analyze their spatial/temporal footprints over sparse versus dense graphs.