b) Find the local maximum and minimum value of $f(x) = 8x^5 - 15x^4 + 10x^2$.

OR

c) State and prove Taylor's Theorem.

d) Test the differentiability of the function :

$$f(x)\begin{cases} x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

at $x = 0$

□□

3rd Sem

2017

Full Marks - 80

Time - 3 hours

The questions are of equal value

Answer ***all*** questions selecting either {(a),(b)} or {(c),(d)} from each question

1. a) Using $\in$-δ approach prove that :

$$\lim_{x \to 0} \frac{\sin x}{x} - 1.$$

b) Let $f : X \to R$ be continuous and let $a \in R$, then prove that $\{x \in X : f(x) > a\}$ is an open subset of X and $\{x \in X : f(x) \geq a\}$ is a closed subset of X.

OR

c) Prove that the image of a closed bounded set under a continuous function is also closed and bounded.

d) Let $f : R \to R$ be additive i.e. $f(x+y) = f(x) + f(y)$. If f(x) is continuous at a point, then prove that $f(x) = ax$, where $a = f(1)$.

[Turn Over

2. a) State and prove intermediate value theorem.

b) Discuss the continuity of

$$f(x) = \begin{cases} 1+x & (-\infty < x < 0) \\ 1+[x]+\sin x & (0 \le x < \pi/2) \\ 3 & (x \ge \pi/2) \end{cases}$$

OR

c) Let X be closed and bounded subset of R and $f: X \to R$ be continuous. Then prove that f attains its maximum and minimum.

d) Let f : [a, b] → R be strictly increasing and continuous on [a, b]. Then prove that f has an inverse $f^{-1} : [f(a), f(b)] \to R$ such that f^{-1} is strictly increasing and continuous on [f(a), f(b)].

3. a) State and prove Generalized mean value theorem.

b) Prove that the function f(x) = |x| is continuous on R but not differentiable at x = 0.

OR

c) State and prove Rulle's Theorem and give its geometrical interpretation.

d) Compute the approximate value of f(17) where $f(x) = \sqrt{x}$.

4. a) State and prove Darboux's Theorem.

b) By L'Hospital's Rule evaluate the following :

i) $\lim_{x \to 0} \frac{1-\cos x}{x^2}$

ii) $\lim_{x \to \infty} x^{1/x}$

OR

c) Show that :

$$\frac{x}{1+x} < \log(1+x) < x \text{ for all } x > 0.$$

d) Show that the function $f(x) = \sin x$, $x \in [0, \pi/4]$ is approximated by a polynomial $p(x) = x - x^3/6$ with an error less than $\frac{1}{400}$.

5. a) State and prove Cauchy's mean value theorem. Also prove that $x < \tan x$ $(0 < x < \pi/2)$

[Turn Over

5. a) Let ϕ be a homomorphism of G onto $\overline{G}$ with Kernel K. Then prove that $\frac{G}{K} \approx \overline{G}$. 10

b) i) If ϕ is a homomorphism of G into $\overline{G}$, then prove that

a) $\phi(e) = \overline{e}$, the unit element of $\overline{G}$

b) $\phi(x^{-1}) = \phi(x)^{-1}$ for all $x \in G$. 3

ii) Let G is the group of nonzero real numbers under multiplication, $\overline{G} = G$. $\phi(x) = x^2$ all $x \in G$. Prove that ϕ is homomorphism. Determine the Kernel. 3

OR

c) Prove that every group is isomorphic to a subgroup of A(S) for some appropriate S. 10

d) i) Let G be the group of all real 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $ad - bc \neq 0$, under matrix multiplication. Let $\overline{G}$ be the group of all

b) If G is a ginite group whose order is a prime number p, then prove that G is a cyclic group. 5

c) i) If H is a subgroup of G and $a \in G$, let

$$aHa^{-1} = \{aha^{-1} / h \in H\}.$$

Show that aHa^{-1} is a subgroup of G. 3

ii) Define a cyclic group and given an example of cyclic group. 3

OR

d) If H and K are finite subgroup of G, of order 0(H) and 0(K), respectively, then prove that

$$0(HK) = \frac{0(H)0(K)}{0(H \cap K)}$$ 10

e) i) Prove that any subgroup of a cyclic group is itself a cyclic group. 3

ii) If G is a finite group and $a \in G$, then prove that $a^{0(G)} = e$. 3

3. a) State and prove Lagrange's theorem. Give an example that its converse is not true. 10

b) i) Determine whether the following permutation is odd or even.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 4 & 1 & 5 \end{pmatrix}$$ 3

ii) Prove that for all $a \in G$,

$$Ha = \{x \in G \mid a \equiv x \bmod H\}$$ 3

OR

c) Prove that if HK is a subgroup of G if and only if HK = KH. 10

d) i) Prove that permutation is the product of its cycles. 3

ii) Prove that if p is a prime number and a is any integer, then $a^p \equiv a \bmod p$. 3

4. a) Define normal subgroup of a group. Prove that the subgroup N of G is a normal subgroup of G if and only if every left coset of N in G is a right coset of N in G. 10

b) i) Show that every subgroup of an abelian group if normal. 3

ii) If G is a finite group and N is a normal subgroup of G, then prove that

$$0\left(\frac{G}{N}\right) = \frac{0(G)}{0(N)}$$ 3

OR

c) State and prove Cauchy's Theorem for abelian group. 10

d) i) If G is a group and H is a subgroup of index 2 in G. Prove that H is a normal subgroup of G. 3

ii) Prove that N is a normal subgroup of G if and only if $gNg^{-1} = N$ for every $g \in G$. 3